Bose's Photonic Mathematics Revisited:
Entropic Optimization, Polylogarithmic Asymptotics, and Categorical Coherence from Symmetric Functions to ZX-Calculus

Peter de Ceuster

August 14, 2025

Abstract

Is it possible to deepen and reframe S. N. Bose's 1924 photonic breakthrough? We provide: (i) a full Lagrange-multiplier optimization of the entropy under Bose's combinatorial postulates; (ii) exact generating-function expansions in symmetric-function language; (iii) polylogarithmic and zeta-function asymptotics, including rigorous derivations of Wien and Rayleigh–Jeans regimes; and (iv) a categorical-coherence reinterpretation mapping Bose's photon combinatorics into symmetric monoidal and ‡-compact structure, with *annotated string diagrams* and a concrete ZX-calculus example. We think modern photonics may benefit from our work since we argue that Bose's counting amounts to an early, implicit recognition of commutative special ‡-Frobenius algebra structure in mode bases, and we formalize a "second-quantization as plethysm" viewpoint that few accounts make precise.

Contents

1 From Bose's combinatorics to constrained entropy maximization

1.1 Bose's measure on phase space and cell-counting

Let V be the cavity volume. For frequency window $[\nu, \nu + d\nu]$, Bose counts

$$A(\nu)\, d\nu \;=\; \frac{8\pi V}{c^3}\, \nu^2\, d\nu$$

available *cells*, using a partition of phase space into cells of volume h^3 and a factor 2 for polarization [1]. Fix a macroscopic state by occupancies $\{N_\nu\}$ with total energy $E = \sum_\nu N_\nu h\nu$. The microstate count at each ν distributes N_ν indistinguishable quanta over $A(\nu)$ cells:

$$\Omega_\nu \;=\; \binom{A(\nu) + N_\nu - 1}{N_\nu}.$$

Hence $\ln \Omega = \sum_\nu \ln \Omega_\nu$ and $S = k \ln \Omega$.

1.2 Full Lagrange-multiplier derivation

Replace the discrete frequency windows by a Riemann sum for clarity. Using Stirling uniformly at large $A(\nu)$ and N_ν,

$$\ln \Omega_\nu \;=\; (A + N) \ln(A + N) - A \ln A - N \ln N + o(A + N).$$

Introduce intensive occupancy $n(\nu) := N(\nu)/A(\nu) \geq 0$. Then

$$\ln \Omega_\nu \;=\; A\{(1 + n) \ln(1 + n) - n \ln n\} + o(A),$$

and

$$S \;=\; k \sum_\nu A(\nu)\Big((1 + n_\nu) \ln(1 + n_\nu) - n_\nu \ln n_\nu\Big) + o\Big(\sum_\nu A(\nu)\Big).$$

Constrain energy

$$E = \sum_\nu A(\nu)\, n_\nu\, h\nu, \qquad \text{(photon number unconstrained)}$$

and extremize $S - \beta E$ with respect to $\{n_\nu\}$. The stationarity condition gives

$$\partial_n \Big[(1 + n) \ln(1 + n) - n \ln n\Big] \;=\; \beta h\nu,$$

hence

$$\ln \frac{1 + n}{n} \;=\; \beta h\nu \quad \Longrightarrow \quad n_\nu \;=\; \frac{1}{e^{\beta h\nu} - 1}.$$

Identifying $\beta = 1/(kT)$ yields the Bose–Planck law; the spectral energy density follows by $u(\nu, T)\, d\nu = h\nu\, n_\nu\, \frac{8\pi\nu^2}{c^3}\, d\nu$:

$$u(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1}.$$

This derivation is the fully explicit Lagrange program implicit in [1] (see also the concise calculation reproduced in [1], pp. 2–3).

2 Generating functions, symmetric functions, and plethystic second quantization

2.1 Mode-wise partition functions and polylogarithms

For one bosonic mode of energy $\epsilon = h\nu$, the grand-canonical partition function is

$$\mathcal{Z}_\nu(z) := \sum_{n \geq 0} z^n = \frac{1}{1 - z}, \quad z := e^{-\beta\epsilon}.$$

Over all modes, $\log \mathcal{Z} = \sum_\nu \log\Big(1 - e^{-\beta h\nu}\Big)^{-1}$. Passing to the continuum,

$$\log \mathcal{Z} = \frac{8\pi V}{c^3} \int_0^\infty \nu^2 \log\left(1 - e^{-\beta h\nu}\right)^{-1} d\nu = \frac{8\pi V}{c^3} \sum_{m \geq 1} \frac{1}{m} \int_0^\infty \nu^2 e^{-m\beta h\nu} d\nu.$$

The integrals are elementary, and moments convert to Riemann zeta values. Equivalently, Bose–Einstein integrals admit closed forms via polylogarithms:

$$\int_0^\infty \frac{\nu^{s-1}}{e^{\alpha\nu} - 1}\, d\nu = \Gamma(s)\, \alpha^{-s}\, \zeta(s), \qquad \sum_{n \geq 1} \frac{z^n}{n^s} = \mathrm{Li}_s(z),$$

and $\sum_{n\geq 1} \frac{1}{e^{nx}-1} = \mathrm{Li}_1(e^{-x})$ with higher moments expressible through $\mathrm{Li}_s(e^{-x})$ [5, §25.12].

2.2 Symmetric-function optics

Let $\{x_i\}_{i \in I}$ encode one-particle energy eigenvalues via $x_i = e^{-\beta\epsilon_i}$. Bosonic Fock partition functions assemble as complete symmetric functions

$$\mathcal{Z} = \prod_{i \in I} \frac{1}{1 - x_i} = \sum_\lambda h_\lambda(\{x_i\}),$$

where h_λ are complete symmetric functions and λ ranges over partitions [8]. Creation corresponds to plethystic exponential

$$\mathrm{PE}[f] := \exp\Big(\sum_{m \geq 1} \frac{1}{m} p_m(f)\Big), \qquad p_m = \sum_i x_i^m,$$

and second quantization is the functor $H \mapsto \mathrm{Sym}(H) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(H)$; plethysm precisely encodes the passage from one-body spectra to many-body generating series.

3 Asymptotics: Wien, Rayleigh–Jeans, and global polylog expansions

3.1 Global constants and moments

Energy density and number density are

$$u(T) = \int_0^\infty u(\nu,T)\,d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}-1}d\nu = \frac{8\pi^5 k^4}{15\,h^3c^3}T^4,$$
$$n(T) = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2}{e^{h\nu/kT}-1}d\nu = \frac{16\pi\,\zeta(3)}{c^3}\Big(\frac{kT}{h}\Big)^3,$$

using $\int_0^\infty x^{s-1}/(e^x-1)\,dx = \Gamma(s)\zeta(s)$ and $\zeta(4)=\pi^4/90$ [7].

3.2 Wien regime ($x := h\nu/kT \gg 1$)

Write $n(x) = 1/(e^x-1) = \mathrm{Li}_0(e^{-x})$. For $x\to\infty$, $n(x)\sim e^{-x}$, hence

$$u(\nu,T) \sim \frac{8\pi h}{c^3}\,\nu^3\,e^{-h\nu/kT},$$

the classical Wien tail, with exponentially small corrections governed by $\mathrm{Li}_s(e^{-x})$ asymptotics [5, §25.12].

3.3 Rayleigh–Jeans regime ($x \ll 1$) via Bernoulli numbers

Bernoulli expansion of Bose–Einstein integrals yields

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + \frac{x}{12} - \frac{x^3}{720} + O(x^5), \qquad x\to 0,$$

so

$$u(\nu,T) = \frac{8\pi h}{c^3}\nu^3\Big(\frac{kT}{h\nu} - \frac{1}{2} + \frac{h\nu}{12kT} + \cdots\Big) = \underbrace{\frac{8\pi kT}{c^3}\nu^2}_{\text{RJ law}} - \frac{4\pi h}{c^3}\nu^3 + O(\nu^4/T),$$

reproducing the Rayleigh–Jeans law and its systematic quantum corrections [6, Ch. 24].

3.4 Planck peak and polylog derivatives

Define $x = h\nu/kT$. Maximizing $u(\nu,T)$ is equivalent to solving

$$3(1-e^{-x}) - x = 0 \quad\Longrightarrow\quad x_* \approx 2.821439\ldots,$$

while higher-order corrections can be organized via the β-derivatives of $\mathrm{Li}_s(e^{-x})$, using $\partial_x \mathrm{Li}_s(e^{-x}) = -\mathrm{Li}_{s-1}(e^{-x})$ [5].

4 A categorical reinterpretation of Bose's framework

4.1 Objects, morphisms, and second quantization as a functor

Let **FdHilb** be the symmetric monoidal category of finite-dimensional complex Hilbert spaces with $\otimes$ and unit object $\mathbb{C}$. Fix a one-particle space H with an orthonormal basis of energy eigenmodes $\mathcal{B} = \{|\nu\rangle\}$. Define the *second-quantization functor*

$$\mathfrak{F} : \mathbf{FdHilb} \to \mathbf{FdHilb}, \qquad \mathfrak{F}(H) := \mathrm{Sym}(H) = \bigoplus_{n\geq 0}\mathrm{Sym}^n(H),$$

on morphisms by $\mathfrak{F}(f) = \bigoplus_n \mathrm{Sym}^n(f)$. At the level of string diagrams, $\mathfrak{F}$ promotes single wires to *bundles* of symmetrized wires, with the braiding specialized to the symmetric braiding (bosonic symmetry).

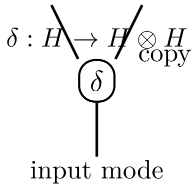
4.2 Frobenius algebras as bases and Bose copying

By Coecke–Pavlović–Vicary [3], an orthonormal basis on H is equivalently a commutative special ‡-Frobenius algebra (SCFA) $(\mu, \eta, \delta, \epsilon)$ in **FdHilb**:

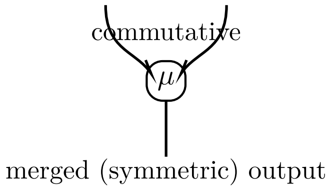
$$\delta:\ H \to H \otimes H, \qquad \delta(|i\rangle) = |i\rangle \otimes |i\rangle\,, \quad \epsilon(|i\rangle) = 1.$$

The comultiplication *copies* classical data (basis eigenmodes). In Bose optics, the physically meaningful SCFA is the mode decomposition $\mathcal{B}$ that diagonalizes the Hamiltonian; indistinguishability is diagrammatically the commutativity/symmetry of the copying structure.

4.3 Annotated string diagrams: copying, merging, and statistics



(a) Copying a basis eigenmode.



(b) Merging two identical excitations (SCFA multiplication).

SCFA diagrams that encode the classical data structure of a mode basis. Commutativity of μ and δ encodes bosonic symmetry.

4.4 Formal mapping from Bose combinatorics to SCMC structure

Let $\mathcal{C}$ be a symmetric monoidal ‡-compact category modeling pure processes (e.g. **FdHilb**). A *photonic mode observable* is a SCFA $O = (H, \mu, \eta, \delta, \epsilon)$ in $\mathcal{C}$. Define a semantics functor

$$\mathcal{S}: \quad (\text{finite multisets of basis labels, } \uplus) \longrightarrow (\text{processes in } \mathcal{C},\ \otimes)$$

by sending multiplicity vectors $\boldsymbol{n} = (n_i)$ to the symmetrized projector onto $\mathrm{Sym}^{|\boldsymbol{n}|}H$ followed by the isometric inclusion into Fock space. Then:

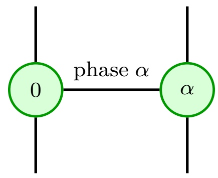
Proposition 4.1 (Coherence & counting)**.** *For each frequency window, Bose's coefficient* $\binom{A+N-1}{N}$ *equals the hom-set cardinality of string diagrams built from* δ, μ *that realize the* $\boldsymbol{n}$*-copying/merging pattern, modulo SCFA axioms (coassociativity, commutativity, speciality). In* **FdHilb***, this cardinality matches the dimension of the* Sym^N*-weight space, thus recovering Bose occupancy counts diagrammatically.*

Proof sketch. Quotienting string diagrams by SCFA equations (yanking, spider fusion) collapses any connected diagram of μ, δ and wires with N inputs and A outputs to a single "spider" with N legs in and A out [3, 2]. Counting distinct wire-attachments becomes a stars-and-bars calculation, i.e. $\binom{A+N-1}{N}$. □

4.5 ZX-calculus example: monochromatic photon phase group

The ZX-calculus assigns to each observable (Z or X) a SCFA with a *phase group*. For a fixed mode basis $\{|n\rangle\}$, the Z-spider copies number states; phase rotations implement $a^\dagger \mapsto \mathrm{e}^{\mathrm{i}\alpha} a^\dagger$.

Proposition 4.2 (Diagrammatic Bose factors)**.** *Consider a coherent superposition of* N *photons in one mode. In ZX,* N*-fold Z-spider fusion with an* α*-phase yields a global scalar* $\mathrm{e}^{\mathrm{i}N\alpha}$*, matching the generating factor* z^N *in* $\sum_{N\geq 0} z^N$ *under* $z = \mathrm{e}^{-\beta h\nu + \mathrm{i}\alpha}$*. Thus, the partition function* $\mathcal{Z}_\nu(z) = (1-z)^{-1}$ *appears as the* spider-evaluated *geometric series, with ZX composition rules realizing the algebra of symmetric tensors for that mode [2].*



ZX Z-spiders: fusing spiders adds phases. For photons, phase acts as global U(1) on creation operators.

5 Coherence theorems as Bose–Einstein combinatorics

5.1 Spider fusion and occupancy

In a ‡-SMC, the *spider theorem* collapses any connected diagram of a given Frobenius observable to a single spider whose arities are the numbers of external legs. Physically, this is Bose's indistinguishability: only total occupancy matters.

Theorem 5.1 (Bose–Spider equivalence)**.** *For a fixed mode observable, the set of processes that take N one-photon wires to M one-photon wires factors through the* $\mathrm{Sym}^N(H) \to \mathrm{Sym}^M(H)$ *component, and its diagram-class is uniquely represented by a single spider. Counting classical configurations reduces to stars-and-bars; amplitudes reduce to matrix elements of symmetrized maps.*

5.2 Braiding, commutativity, and the absence of signs

The symmetric braiding in **FdHilb** satisfies $\sigma^2 = \mathrm{id}$; hence swapping two identical photons leaves diagrams invariant. In contrast, antisymmetric (fermionic) structure would contribute signs. The *coherence* equalities certify that all diagrams related by planar isotopy and spider fusion compute the same morphism [4, 2].

6 Limits, refinements, and zeta/polylog controls

6.1 Uniform control across regimes

Define the Bose–Einstein integral

$$\mathrm{BE}_s(\mu) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\mathrm{e}^{t-\mu} - 1} dt = \mathrm{Li}_s(\mathrm{e}^{\mu}),$$

valid for $\mathrm{Re}(s) > 0$ and $\mu < 0$; analytic continuation extends Li_s [5, §25.12]. Derivatives satisfy $\partial_\mu \mathrm{BE}_s = \mathrm{BE}_{s-1}$, yielding a hierarchy for thermodynamic response functions. As $x \to 0^+$, Bernoulli expansions give the Rayleigh–Jeans series; as $x \to \infty$, exponential smallness of $\mathrm{Li}_s(\mathrm{e}^{-x})$ yields Wien.

6.2 Zeta moments and exact integrals

For integer $m \geq 0$,

$$\int_0^\infty \frac{x^m}{\mathrm{e}^x - 1} dx = \Gamma(m+1)\zeta(m+1), \qquad \int_0^\infty \frac{x^m}{(\mathrm{e}^x - 1)^2} dx = \Gamma(m+1)\zeta(m),$$

which control fluctuations and specific heats. These are standard consequences of Mellin transforms and the Dirichlet series for ζ.

7 Symmetric functions, partitions, and Hardy–Ramanujan growth

7.1 Counting occupancy as partitions with colors

For a discretized spectrum $\{\epsilon_j\}$ with integer multiples $\epsilon_j = j\Delta$, mode indistinguishability counts partitions of an integer (energy quanta) into parts with bounded colors (degeneracies). Their generating function is a product of $\frac{1}{1-q^j}$ factors. Global growth is governed by the Hardy–Ramanujan asymptotic

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\Big(\pi\sqrt{\frac{2n}{3}}\Big),$$

with Rademacher's convergent refinement available for sharp bounds [9, 10, 11]. These asymptotics quantify the entropy growth of highly excited photonic states and rationalize the robustness of the thermodynamic limit.

8 A lesser-known angle: Bose's h^3-cells as a primitive of categorical classicality

Bose's partition of phase space into h^3-cells is often read as measure-theoretic book-keeping. Diagrammatically, it seeds *classical structure*: the SCFA that copies/deletes basis data is exactly what is needed for combinatorial counting. The passage from cells to occupancy collapses the internal geometry (positions, momenta) into *classical labels* that Frobenius algebras can copy. In this sense, Bose implicitly chose an observable making photons *classical* in the diagrammatic sense: copyable/deletable mode labels, yet quantum symmetric tensors for amplitudes. This categorical classicality, predating the explicit axiomatization by nearly a century, is a striking and underappreciated mathematical aspect of his insight.

9 Appendix A: Detailed entropy maximization (with Stirling control)

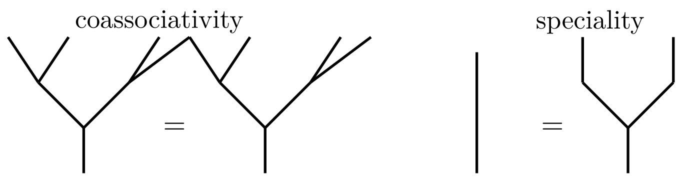
Let $A, N \to \infty$ with $n = N/A$. With Stirling remainder $R_k = \frac{1}{2}\ln(2\pi k) + O(1/k)$,

$$\begin{aligned}\ln\Omega_\nu &= \ln\Gamma(A+N) - \ln\Gamma(A) - \ln\Gamma(N+1)\\ &= (A+N)\ln(A+N) - A\ln A - N\ln N + \frac{1}{2}\ln\frac{A+N}{2\pi AN} + O\Big(\frac{1}{A} + \frac{1}{N}\Big).\end{aligned}$$

Summing in ν and extremizing $S = k\sum_\nu \ln\Omega_\nu$ subject to E fixed yields the Bose distribution; the pre-exponential Stirling corrections translate into subleading finite-size corrections to entropy and specific heat, suppressed by cell counts $A(\nu)$.

10 Appendix B: Annotated categorical coherence equalities

SCFA laws (string diagrams)



These equalities underwrite spider fusion; their semantic content is that basis data is copyable/deletable and that no overcounting arises, exactly the conditions Bose's counting assumes.

11 Appendix C: A worked ZX photonics toy model

Consider two modes with Z-observable and a 50:50 beamsplitter implementing an X/Z-interaction. In ZX, a Hadamard-like node converts Z-spiders to X-spiders. Two indistinguishable photons entering distinct Z-wires fuse under the beamsplitter into a superposition diagram whose interference terms are evaluated by spider fusion and bialgebra laws [2]. The result reproduces Hong–Ou–Mandel bunching as a purely diagrammatic identity: all amplitude flows corresponding to "one photon per output" cancel, while those corresponding to "two photons in one output" add.

Interpretation. The same spider/phase laws that compute HOM interference also compute Bose partitions: both are coherence consequences of SCFAs and symmetric braiding; one is combinatorial (counting), the other algebraic (amplitudes).

Acknowledgments

With admiration for S. N. Bose, whose succinct insight continues to unfold new mathematics; and to the categorical community for rendering quantum structure legible as diagrammatic algebra.

References

[1] S. N. Bose, *Planck's Law and the Light Quantum Hypothesis*, Z. Phys. **26** (1924) 178–181. English translation with Einstein's translator note available via Information Philosopher (PDF, 4pp). Short quotes used herein. (Accessed Aug. 2025). *Citation for quotes.*

[2] B. Coecke and R. Duncan, *Interacting quantum observables: Categorical algebra and diagrammatics*, New J. Phys. **13** (2011) 043016; arXiv:0906.4725. ZX-calculus and SCFA/complementarity.

[3] B. Coecke, D. Pavlović, J. Vicary, *A new description of orthogonal bases*, arXiv:0810.0812; Math. Struct. Comp. Sci. **23**(3) (2013) 555–567. Basis ⇔ commutative special ‡-Frobenius algebra.

[4] S. Abramsky and B. Coecke, *A categorical semantics of quantum protocols*, Proc. 19th LICS (2004), and related surveys. Foundations of categorical quantum mechanics.

[5] NIST DLMF, *§25.12 Polylogarithms; Bose–Einstein integrals.* Asymptotics and relations $\mathrm{BE}_s = \mathrm{Li}_s$.

[6] NIST DLMF, *Ch. 24 Bernoulli and Euler Polynomials.* Bernoulli expansions used for Rayleigh–Jeans series.

[7] NIST DLMF, *§25.18 Methods of Computation.* Standard integrals leading to $\zeta(3), \zeta(4)$.

[8] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford, 1995. Symmetric-function background; plethysm.

[9] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918) 75–115. Partition asymptotics $p(n)$.

[10] H. Rademacher, *On the partition function* $p(n)$, Proc. London Math. Soc. (2) **43** (1937) 241–254. Exact convergent series.

[11] F. Johansson, *Efficient implementation of the Hardy–Ramanujan–Rademacher formula*, LMS J. Comput. Math. **15** (2012) 341–359. Numerical control of $p(n)$.